

Matrix division is the opposite of multiplication.

→ Many problems can be written as matrix division.

EX: Solve the system  $\begin{cases} 2x + 4y = 8 \\ -3x + 4y = -2 \end{cases}$

This is equivalent to  $\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

so  $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 8 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}}$  Matrix Division!

Note: We must be careful about which side we are dividing on.

$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \xrightarrow{\text{Divide on left}} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 8 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}}$

(usually we want to divide on left)

$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 12 \end{bmatrix}$

$\begin{bmatrix} x & y \end{bmatrix} = \frac{\begin{bmatrix} -4 & 12 \end{bmatrix}}{\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}}$  Divide on right

Just as with multiplication, we compute division by converting to a system of equations.

(This is usually a very bad method...)

EX: Divide  $\frac{\begin{bmatrix} 8 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}}$

It is equivalent to solve

$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \rightsquigarrow \begin{cases} 2x + 4y = 8 \\ -3x + 4y = -2 \end{cases}$

(As children, we learned to solve this by adding equations to eliminate a variable and then substituting....)

$$\begin{array}{r} 2x + 4y = 8 \\ -(-3x + 4y = -2) \\ \hline 5x = 10 \end{array} \Rightarrow x = 2$$
  
 subst.  $2(2) + 4y = 8 \Rightarrow y = 1$  }  $\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So  $\frac{\begin{bmatrix} 8 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Check:  $\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 + 4 \\ -6 + 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

Problem: Messing about with equations quickly becomes difficult when you have more than two variables and/or more than two equations.

Solution #1: Instead of converting to a system of equations and adding multiples of equations together, convert to an augmented matrix and use "Gaussian Elimination"

$$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix} \xrightarrow{\text{divide on left}} \begin{bmatrix} 2 & 4 & | & 8 \\ -3 & 4 & | & -2 \end{bmatrix} \rightarrow \dots \text{etc.}$$

(You might have learned this as a child.)

We will instead learn to divide using a method which is more powerful and more interesting.

Solution #2: Factor the matrix into its "LU-Decomposition" and use this to compute matrix division.

The idea for the LU-Decomposition comes from the following

FACT: Some matrices are easier to divide by than others.

→ Instead of doing one hard division, we will do two easy divisions!

Def: The main diagonal of a matrix starts at the upper-left corner and moves diagonally

EX: 
$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix}$$
 Main Diagonal

EX: 
$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$
 Main Diagonal

EX: 
$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$$
 Main Diagonal

Def: An upper-triangular ("upper Δ") matrix is 0 below the main diagonal.

A lower-triangular ("lower Δ") matrix is 0 above the main diagonal.



EX  $\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Upper-Triangular

EX:  $\begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$

Upper-Triangular

EX:  $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

Lower-Triangular

EX:  $\begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}$

Lower-Triangular

Dividing by triangular matrices can be done using (forward or back)-substitution (except for one "special case" to be discussed later)

EX: Divide  $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

To solve this we will convert to equations:

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} 2x = 6 \\ 3x + 4y = 5 \\ -x + 2y + 3z = 1 \end{cases}$$

First equation determines x.

→ Plug x value into next equation to get y.

→ Plug x & y values into next equation to get z.

$$\begin{cases} 2x = 6 \Rightarrow x = 3 \\ 3x + 4y = 5 \Rightarrow 3 \cdot 3 + 4y = 5 \Rightarrow y = -1 \\ -x + 2y + 3z = 1 \Rightarrow -3 + 2(-1) + 3z = 1 \Rightarrow z = 2 \end{cases}$$

So  $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

(This is called "forward substitution")

Check:  $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9-4 \\ -3-2+6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

Actually what is needed is not  $\Delta$  shape but that the last nonzero element in lower rows is further to the right.

→ The matrix should have a "stair-step" shape

$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 4 & 3 \end{bmatrix}$   
"stair-step" shape

EX: Divide  $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix} \rightsquigarrow \begin{cases} 2x + 3y - z = 2 \\ 4y + 2z = -2 \\ 3z = -9 \end{cases}$$

(Start with the last equation & last variable  
 — substitute upwards ("back-substitution"))

$$\begin{aligned} 3z &= -9 \rightsquigarrow z = -3 \\ 4y + 2z &= -2 \rightsquigarrow 4y + 2(-3) = -2 \\ &\phantom{4y + 2z} \rightsquigarrow 4y - 6 = -2 \\ &\phantom{4y + 2z} \rightsquigarrow 4y = 4 \\ &\phantom{4y + 2z} \rightsquigarrow y = 1 \\ 2x + 3y - z &= 2 \rightsquigarrow 2x + 3(1) - (-3) = 2 \\ &\phantom{2x + 3y - z} \rightsquigarrow 2x + 3 + 3 = 2 \\ &\phantom{2x + 3y - z} \rightsquigarrow 2x + 6 = 2 \\ &\phantom{2x + 3y - z} \rightsquigarrow 2x = -4 \\ &\phantom{2x + 3y - z} \rightsquigarrow x = -2 \end{aligned}$$

So  $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

Check:  $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 + 3 + 3 \\ 4 - 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix}$

LU- Decomposition.

All matrices can be factored as a product  
 $A = L \cdot U$

where  $U$  is an upper-triangular matrix  
 with the same size as  $A$

$L$  is a lower-triangular matrix  
 which is square ( $\#rows(L) = \#cols(L) = \#rows(A)$ )  
 with all 1 on the diagonal

→ Knowing  $L$  &  $U$  is like knowing  $\frac{1}{A} \dots$

(Computing  $L$  &  $U$  is not difficult. But  
 first I will give an example showing  
 how to use  $A = L \cdot U$  to divide.)

EX: Divide  $\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$

if  $\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

$A \qquad \qquad \qquad L \qquad \qquad \qquad U$

(Instead of dividing by  $A$ , divide by  $L$  &  $U$ )



$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$$

Convert from A to LU decomp.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$$

Divide by left-most matrix (L) first.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$$

Convert to multiplication to get the answer.  
 Note: Do not use  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  because that is final answer

$a = -1$   
 $b = -5$   
 $c = 4$

Now divide by right-most matrix (U).

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 4 \end{bmatrix}$$

Convert to multiplication to get the answer.

(back-subst.)  
 $z = -2$   
 $y = 1$   
 $x = 3$

Answer:  $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

Check:  $\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6-1-6 \\ 12-3-16 \\ -6+6 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$

EX: Divide  $\begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -5 & 6 \\ 3 & -3 & 0 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -7 \end{bmatrix}$

if  $\begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -5 & 6 \\ 3 & -3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

A L U

Solve  $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -7 \end{bmatrix}$

Divide by L  $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -7 \end{bmatrix}$

$a = 3$   
 $b = -2$   
 $c = -6$

Divide by U  $\begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -6 \end{bmatrix}$

Back substitution:  $3z = -6 \Rightarrow z = -2$   
 $-2x + y = -2 \Rightarrow y = \text{free}$   
 $x = 1 + \frac{1}{2}y$

$3w + x - 2y + 2z = 3$   
 $3w + (1 + \frac{1}{2}y) - 2y + 2(-2) = 3$   
 $w = 2 + \frac{1}{2}y$

Answer:  $\begin{bmatrix} 2 + \frac{1}{2}y \\ 1 + \frac{1}{2}y \\ y \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} y$